ISRAEL JOURNAL OF MATHEMATICS, Vol. 53, No. 2, 1986

ON WEAKLY NEIGHBORLY POLYHEDRAL MAPS OF ARBITRARY GENUS

BY

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ABSTRACT

A weakly neighborly polyhedral map (w.n.p. map) is a 2-dimensional cell-complex which decomposes a closed 2-manifold without boundary, such that for every two vertices there is a 2-cell containing them. We lay the foundation for an investigation of the w.n.p. maps of arbitrary genus. In particular we find all the w.n.p. maps of genus 0, we prove that for every $g > 0$ the number of w.n.p. maps of genus g (orientable or not) is finite, and we find an upper bound for the number of vertices in a w.n.p. map of genus $g > 0$. This upper bound grows as $(4g)^{2/3}$ when $g \to \infty$.

1. Introduction

A (topological) *polyhedral map* is a 2-dimensional topological cell-complex which decomposes a closed connected 2-manifold without boundary. Where no confusion is likely to arise, we will not distinguish between the polyhedral map and the manifold decomposed by it. Thus we say that the polyhedral map is of genus g if the 2-manifold decomposed by it is of genus g, etc. We define the genus g as $g = \frac{1}{2}(2 - \chi)$, where χ denotes the Euler characteristic. Thus if the map is orientable then g is defined as usual and can be any positive integer or 0, and if the map is not orientable, g may take values of the form $\frac{1}{2}n$, n being any positive integer. The 0-, 1- and 2-dimensional cells (faces) of the polyhedral map are its *vertices*, *edges* and *facets*, respectively.

It follows from the definition that the intersection of every two faces of a polyhedral map is a face (possibly empty) of the map, that every facet is a closed topological disc, which is at least 3-sided, and that the valence of every vertex (that is, the number of edges incident to the vertex) is at least 3. It is also clear that the dual of a polyhedral map is again a polyhedral map. We use the term *polyhedral map* in order to distinguish between this kind of a map and the more

Received February 13, 1984 and in revised form June 26, 1985

general concept of a map used, e.g., in [9], and because the map defined by a polyhedron-like structure in the Euclidean 3-space R^3, with convex facets, has the properties required by our definition.

A polyhedral map M is *geometric* if each facet of M is a convex polygon and no two adjacent facets (that is, facets which share a common edge) lie in the same plane. A polyhedral map M is *geometrically realizable* in the Euclidean n-space R^n if there is a geometric polyhedral map in R^n which is combinatorially isomorphic to M, and if $n = 3$ we say simply that M is *geometrically realizable.*

Of special interest is the class of polyhedral maps in which the edge graph is complete (in such a map every facet is a triangle) or, dually, polyhedral maps in which every two facets share a common edge (and hence each vertex is 3-valent). This class has been studied mainly by Ringel and Youngs (see [12]) as a part of a more general class of maps, in connection with the Heawood Map Color Theorem. Ringel and Youngs also characterized the manifolds on which such a map does exist. Thus it seems natural to generalize this class of polyhedral maps by considering polyhedral maps in which every two vertices belong to a common facet or, dually, polyhedral maps in which every two facets have a non-empty intersection. It is of course sufficient to work with only one of these two families. This gives rise to the following definition.

Define a polyedral map M to be *neighborly* if for every two vertices of M there is an edge of M which contains both, and *weakly neighborly* if for every two vertices of M there is a facet of M which contains both of them. We abbreviate "weakly neighborly polyhedral map" by "w.n.p. map".

By a well-known theorem of Steinitz (see [14] or [11, Section 13.1]), every polyhedral map of genus 0 is geometrically realizable. On the other hand, even for the torus we don't know which are the toroidal polyhedral maps that are not geometrically realizable (see [13]). It is known, however, that for every integer $g > 0$ there are orientable polyhedral maps of genus g that are not geometrically realizable in any R^n (see [1, p. 236] or [6, p. 9]). From Ringel and Youngs' work (see [12]) there follows the existence of an infinite family of neighborly polyhedral maps, but only two of them, the boundary complex of the 3-simplex and the Császár torus (see [10] or [1, p. 217]), are known to be geometrically realizable.

It seems that w.n.p. maps have not yet been studied. The main question is: for a given g, which are the w.n.p. maps of genus g (orientable or not), and which of them are geometrically realizable? For $g = 0$, the (boundary complexes of the) triangular prism and the pyramides are clearly w.n.p. maps, and intuitively one tends to believe that there is no other w.n.p. map of genus 0. (This is indeed the case, as shown in our Theorem 1.) For other values of g the problem is rather difficult, and must be investigated separately for each g.

The main purpose of the present work is to lay the foundations for such an investigation. We prove that (unlike the case $g = 0$) for every $g > 0$ the number of w.n.p. maps of genus g is finite, we find bounds for the number of vertices in a w.n.p. map of genus $g > 0$, and we describe some combinatorial restrictions on the structure of a w.n.p. map of genus g, which should be of help in the detailed investigation of the w.n.p. maps of a given genus g. Using these restrictions, we were able to find (in [8] and [2]) all the w.n.p. maps on the torus — there are just five, and precisely two of them are not geometrically realizable — and to prove (in [3]) that there are no orientable w.n.p. maps of genus 2. In order to facilitate the use of the present work as a reference for further investigations, each of the restrictions mentioned here is given a reference number, beside being a part of some lemma or theorem.

The following will serve as our standard notation. We assume a polyhedral (or w.n.p.) map of genus g with V vertices $v_1, \ldots, v_V$, E edges and F facets $f_1, \ldots, f_F$. For $1 \leqq i \leqq F$, k_i is the number of vertices of F, and we take the notation to be such that $k_1 \geqq k_2 \geqq \cdots \geqq k_F$. The vertices too are labelled according to decreasing order of valences (degrees), that is, $\deg v_1 \geqq \deg v_2 \geqq \cdots \geqq \deg v_V$. $N(v_i)$ and $N(f_i)$ denote the set of vertices adjacent to v_i and the set of facets adjacent to f_i, respectively. (Two vertices are adjacent if they are distinct and belong to the same edge, and two facets are adjacent if they are distinct and share a common edge.) Though $f_i \cap f_j$ is, of course, the intersection of the facets f_i and f_j, we will use $|f_i \cap f_j|$ to denote the number of vertices common to f_i and f_j, where no confusion is likely to arise. p_i denotes the number of i-gonal facets of the map, and $(p_3, p_4, \ldots)$ is the *p-vector* of the map. V_i denotes the number of vertices of valence i, and $(V_3, V_4, \ldots)$ is the *v-vector* of the map. $V(g^+)$ ($V(g^-)$) is the maximal number of vertices that an orientable (non-orientable) w.n.p. map of genus g can have, and $V(g) = \max\{V(g^+), V(g^-)\}$.

In Section 2, after presenting some preparatory lemmas, we prove Theorem 1 which describes all the w.n.p. maps of genus 0. In Section 3 we present some more lemmas which are used in Section 4 to prove the main results of the present work. In Section 4 we prove that, for every $g > 0$, $V(g)$ is bounded, and we describe, in Theorem 8, a (rather complicated) upper bound for it. From this it follows (Theorem 9) that for every $g > 0$ the number of w.n.p. maps of genus g is finite. In Theorem 10 we show that $\overline{\lim}_{g \to \infty} (4g)^{-2/3} V(g) \leqq 1$.

All the relations of Sections 1–4 (with the exception of (9)–(12)) for w.n.p.

maps of genus $g>0$ can be derived from a very small set of equalities and inequalities, namely from (1), (3), (5), (13), (19) and the fact that $k_1 \geqq \cdots \geqq k_F \geqq 3$. These can be regarded as the basic relations from which the others are derived. In Sections 3 and 4 we try to derive the utmost of information from it for the relations between V, g and k_1.

We use the symbols $\lfloor x \rfloor$ and $\lceil x \rceil$ to denote the greatest integer $\leqq x$ and the least integer $\geqq x$, respectively.

2. Weakly neighborly polyhedral maps on the 2-sphere

In this section we prove (Theorem 1) that the only w.n.p. maps on the 2-sphere are the (boundary complexes of the) triangular prism and the pyramides. For that we need some equalities and inequalities which involve the parameters of the map. Those which hold for every polyhedral map are collected in Lemma 1, the others, which refer to w.n.p. maps, in Lemma 2. Most of them will be used frequently in the entire work.

LEMMA 1. *The following hold for every polyhedral map*:

(1) $2V+4g-4=\Sigma_{i=1}^{F}(k_i-2)$,

(2) $k_i \geqq 3$ *for every* $1 \leqq i \leqq F$,

(3) $k_1+k_2+\cdots+k_\mu \leqq V+\mu(\mu-1)$ *for every* $1 \leqq \mu \leqq F$,

(4) $k_1 \leqq V-1$.

PROOF. (1) This is just a reformulation of Euler's equation $V-E+F=2-2g$, since $2E=\Sigma_{i=1}^{F} k_i$.

(2) Trivial.

(3) Each of the facets $f_1,\ldots,f_\mu$ has at most two vertices in common with each of the other facets. The number of pairs of facets among these μ facets is $\frac{1}{2}\mu(\mu-1)$, hence $k_1+\cdots+k_\mu \leqq V+2\cdot\mu(\mu-1)/2=V+\mu(\mu-1)$.

(4) Follows from (3) with $\mu=2$ and (2) with $i=1$. □

LEMMA 2. *The following hold for every w.n.p. map*:

(5) $V(V-1)=\Sigma_{i=1}^{F} k_i(k_i-2)$,

(6) $(V-3)(V-4)-12g=\Sigma_{i=1}^{F}(k_i-2)(k_i-3)$,

(7) $k_1 \geqq \lceil (V+1)/2 \rceil - g$, *and if* $k_1=(V+1)/2-g$ *then* $g=\frac{1}{2}$ *or* $g=1$.

PROOF. (5) Each of the $\frac{1}{2}V(V-1)$ pairs of vertices of the map yields either an edge or a diagonal of a facet. The number of edges is $\frac{1}{2}\Sigma_{i=1}^{F} k_i$ and the number of diagonals in the facet f_i is $\frac{1}{2}k_i(k_i-3)$. Hence

$$\tfrac{1}{2}V(V-1)=\tfrac{1}{2}\left(\sum_{i=1}^{F} k_i+\sum_{i=1}^{F} k_i(k_i-3)\right)=\tfrac{1}{2}\sum_{i=1}^{F} k_i(k_i-2).$$

(6) This is just (5)–3(1) (that is, equality (5) subtracted by three times the equality (1)).

(7) From (1) and (5) we get $V(V-1)=\Sigma_{i=1}^{F} k_i(k_i-2)\leqq k_1\Sigma_{i=1}^{F}(k_i-2)=k_1(2V+4g-4)$. Hence

$$k_1 \geqq \frac{V(V-1)}{2V+4g-4}=\frac{V}{2}-g+\frac{V+4g(g-1)}{2V+4g-4}.$$

Now

$$\frac{V+4g(g-1)}{2V+4g-4}\geqq\frac{1}{2}$$

holds for every g and V, with equality iff $g=\frac{1}{2}$ or $g=1$, and (7) follows. □

Using Lemmas 1, 2 we can now prove:

THEOREM 1. *The only w.n.p. maps of genus 0 are the (boundary complexes of the) pyramides and the triangular prism.*

PROOF. Let M be a w.n.p. map of genus 0. If $V=4$ then M is the (boundary complex of the) 3-simplex, which is indeed neighborly. Thus we assume from now on that $V\geqq 5$. First assume that $k_1+k_2\leqq V+1$, that is, $k_2-3\leqq V-k_1-2$. Now

$$(V-3)(V-4)-(k_1-2)(k_1-3)\underset{(6)}{=}\sum_{i=2}^{F}(k_i-2)(k_i-3)$$

$$\leqq(k_2-3)\sum_{i=2}^{F}(k_i-2)\underset{(1)}{=}(k_2-3)(2V-4-(k_1-2))\leqq(V-k_1-2)(2V-k_1-2).$$

Hence $2k_1^2-k_1(3V+1)+V^2+V-2\geqq 0$. The roots (for k_1) of the left side are $\frac{1}{4}(3V+1\pm\sqrt{(v-1)^2+16})$, hence

$$\text{either}\quad k_1\geqq\left\lceil\frac{3V+1+(V-1)}{4}\right\rceil=V\quad\text{or}\quad k_1<\frac{3V+1-(V-1)}{4}=\frac{V+1}{2}.$$

But the first is impossible by (4), and the second contradicts (7).

Hence $k_1+k_2\geqq V+2$, and together with (3) for $\mu=2$ we get $k_1+k_2=V+2$. Thus f_1 and f_2 share a common edge — denote its vertices by a, b — and $f_1\cup f_2$ contains all the V vertices of the map. Now each of the other facets of M contains at most four vertices (that is, $k_3\leqq 4$) and for every $i\geqq 3$ such that $k_i=4$, f_i shares an edge with f_1 and another edge with f_2. Also there is just one facet beside f_1 and f_2 which contains the vertex a, and that facet is triangular. Similarly there is just one facet other than f_1, f_2 which contains the vertex b, and that facet is triangular. Hence, if α denotes the number of quadrangular facets of M other than f_1, f_2, then $\alpha \leqq k_2 - 3$. (3 edges of f_2 are "occupied" by f_1 and by the two triangles mentioned above.)

Equality (6) yields $V^2 - 7V + 12 = 2\alpha + (k_1 - 2)(k_1 - 3) + (k_2 - 2)(k_2 - 3)$ and, since $k_1 = V + 2 - k_2$, we get $k_2^2 - k_2(V+2) + 3V - 3 + \alpha = 0$. $\alpha \leqq k_2 - 3$ implies $k_2^2 - k_2(V+2) + 3V - 3 + k_2 - 3 \geqq 0$, that is, $k_2^2 - k_2(V+1) + 3V - 6 \geqq 0$. The roots (for k_2) of the left side are $\frac{1}{2}(V + 1 \pm |V - 5|)$. Since $V \geqq 5$, we get $k_2 \leqq 3$ or $k_2 \geqq V - 2$. $k_2 \leqq 3$ implies $k_2 = 3$, $k_1 = V - 1$, and M is therefore a pyramid over a $(V-1)$-gon. If $k_2 \geqq V - 2$ then, by $k_2 \leqq k_1$, we get $k_1 \geqq V - 2$, hence $V + 2 = k_1 + k_2 \geqq 2(V-2)$, implying that $V \leqq 6$. If $V = 5$ then $k_2 = 3$, $k_1 = 4$ and we get again a pyramid. If $V = 6$ then $k_1 = k_2 = 4$, $\alpha = 1$ and M is a triangular prism. □

From now on we shall assume that $g > 0$.

3. Combinatorial restrictions

The combinatorial restrictions on the structure of a w.n.p. map, in the form of equalities and inequalities involving the parameters of the map, which we describe here, will be used in the next section for finding bounds for V in terms of g. Not all of them are independent of each other.

In order to get equalities which hold for all polyhedral maps we introduce two new series of parameters a_j and b_j ($j = 1, \dots, F$) which measure to what extent the polyhedral map fails to be weakly neighborly with respect to the facet f_j. a_j denotes the number of pairs (x, y) where x is a vertex of f_j and y is a vertex not in f_j, such that xy is neither an edge nor a diagonal of M. b_j denotes the number of unordered pairs (x, y) of distinct vertices, none of which is in f_j, such that xy is neither an edge nor a diagonal of M. Obviously $a_j \geqq 0$, $b_j \geqq 0$ for all $j = 1, \dots, F$.

LEMMA 3. *For every polyhedral map*

$$V(V-1) = \sum_{i=1}^{F} k_i(k_i - 2) + 2(a_j + b_j) \qquad \textit{for every } 1 \leqq j \leqq F. \tag{8}$$

PROOF. The proof is the same as for (5) up to the obvious changes, noting that if xy $(x \neq y)$ is neither an edge nor a diagonal of the map, then x or y is not contained in f_j. □

LEMMA 4. *The following are equivalent for a polyhedral map M:*

(a) *M is weakly neighborly,*

(b) *$a_j = b_j = 0$ for all $j \in \{1, \dots, F\}$,*

(c) $a_j = b_j = 0$ *for some* $j \in \{1, \dots, F\}$,
(d) $a_j = 0$ *for all* $j \in \{1, \dots, F\}$.

PROOF. The equivalence of (a), (b), (c) is obvious, using (8). (b) $\Rightarrow$ (d) is obvious, and (d) $\Rightarrow$ (a) is obvious too since each vertex is contained in some facet. □

REMARK. Since $a_j \geqq 0$, $b_j \geqq 0$ and because of Lemma 4, all the equalities of this chapter containing a_j or b_j can be read as equalities for w.n.p. maps or as inequalities for polyhedral maps with a_j and b_j being cancelled.

The next theorem is of crucial value and gives rise to many other restrictions. Recall that $|f_i \cap f_j|$ indicates the number of vertices common to f_i and f_j.

THEOREM 2. *The following holds for every polyhedral map*:

$$k_j(V - k_j) = \sum_{|f_i \cap f_j| = 1} (k_i - 2) + \sum_{|f_i \cap f_j| = 2} (2k_i - 5) + a_j \qquad \text{for every } 1 \leqq j \leqq F. \tag{9}$$

PROOF. Let M be a polyhedral map, and fix $j \leqq F$. Then there are $k_j(V - k_j)$ pairs of vertices (x, y) with $x \in f_j$, $y \notin f_j$. By the definition of a_j there are exactly $k_j(V - k_j) - a_j$ pairs of vertices (x, y) with $x \in f_j$, $y \notin f_j$ and x sees y. ("x sees y" means that there is a facet having x *and* y as vertices.) On the other hand, the exact number of such pairs is $\Sigma_{|f_i \cap f_j| = 1}(k_i - 2) + \Sigma_{|f_i \cap f_j| = 2}(2k_i - 5)$. (If x is the common vertex of f_j, f_l, f_m where $f_m, f_l \in N(f_j)$, then x sees exactly $\Sigma_{f_i \cap f_j = x}(k_i - 2) + (k_l - 2) + (k_m - 3)$ vertices that are not in f_j.) Thus

$$k_j(V - k_j) = \sum_{|f_i \cap f_j| = 1} (k_i - 2) + \sum_{|f_i \cap f_j| = 2} (2k_i - 5) + a_j.$$ □

In the next lemma we state two other useful equalities that are equivalent to Theorem 2.

LEMMA 5. *The following hold for every polyhedral map*:

$$\sum_{|f_i \cap f_j| = 2} k_i - \sum_{f_i \cap f_j = \varnothing} (k_i - 2) = (k_j - 2)(V - k_j + 2) + 6 - 4g - a_j \text{ for every } 1 \leqq j \leqq F, \tag{10}$$

$$\sum_{|f_i \cap f_j| = 2} (k_i - 3)^2 + \sum_{|f_i \cap f_j| = 1} (k_i - 2)^2 + \sum_{f_i \cap f_j = \varnothing} k_i(k_i - 2) = (V - k_j)(V - k_j - 1) - 2b_j \tag{11}$$

for every $1 \leqq j \leqq F$.

PROOF. (10) and (9) can be rephrased as

$$k_j(V-k_j)=\sum_{i=1}^{F}(k_i-2)+\sum_{|f_i\cap f_j|=2}(k_i-3)-\sum_{f_i\cap f_j=\varnothing}(k_i-2)-(k_j-2)+a_j.$$

By (1), the right side equals $2V+4g-2+\Sigma_{|f_i\cap f_j|=2}(k_i-3)-\Sigma_{f_i\cap f_j=\varnothing}(k_i-2)-k_j+a_j$. Since $\Sigma_{|f_i\cap f_j|=2}(k_i-3)=\Sigma_{|f_i\cap f_j|=2}k_i-3k_j$, we get

$$k_j(V-k_j)=2V+4g-2-4k_j+\sum_{|f_i\cap f_j|=2}k_i-\sum_{f_i\cap f_j=\varnothing}(k_i-2)+a_j,$$

from which (10) follows.

Regarding (11), by adding twice the equality (10) to the sum of equality (8) and twice equality (1), after reversing the sides of (1), we get

$$\begin{aligned}V^2-V+2\sum_{i=1}^{F}(k_i-2)+2\sum_{|f_i\cap f_j|=2}k_i-2\sum_{f_i\cap f_j=\varnothing}(k_i-2)\\=\sum_{i=1}^{F}k_i(k_i-2)+2(k_j-2)(V-k_j+2)+4V+4+2b_j\end{aligned}$$

which yields

$$\begin{aligned}&V^2-V+2k_j^2-2Vk_j-8k_j+4-2b_j\\&=\sum_{i=1}^{F}(k_i-2)^2+2\sum_{f_i\cap f_j=\varnothing}(k_i-2)-2\sum_{|f_i\cap f_j|=2}k_i\\&=\sum_{|f_i\cap f_j|=1}(k_i-2)^2+\sum_{|f_i\cap f_j|=2}(k_i^2-6k_i+4)+\sum_{f_i\cap f_j=\varnothing}k_i(k_i-2)+k_j^2-4k_j+4.\end{aligned}$$

Now $\Sigma_{|f_i\cap f_j|=2}(k_i^2-6k_i+4)$ has k_j terms, hence it equals $\Sigma_{|f_i\cap f_j|=2}(k_i-3)^2-5k_j$ and (11) follows. □

In the next lemma we rework some of the equalities of Lemma 5. Losing some information (as equalities turn into inequalities), we gain some simplicity. In particular, in (14) only the three parameters V, g and k_1 are involved.

LEMMA 6. *The following hold for every w.n.p. map*:

(12) $$\sum_{f_i\in N(f_j)}k_i\geqq(k_j-2)(V-k_j+2)+6-4g \qquad \textit{for every } 1\leqq j\leqq F,$$

(13) $$\sum_{i=2}^{k_1+1}k_i\geqq(k_1-2)(V-k_1+2)+6-4g,$$

(14) $$(k_1-2)(V-2k_1)+2\leqq 4g.$$

PROOF. (12) follows from (10). (13) is obtained from (12) by taking there

$j = 1$, and recalling that $k_1 \geqq k_2 \geqq \cdots \geqq k_F$. The last fact also implies that $k_1^2 \geqq \sum_{i=2}^{k_1+1} k_i$. From this inequality and (13) we obtain (14). □

LEMMA 7. *The following hold for every polyhedral map*:

$$\sum_{i=2}^{k_1+1} (k_i - 3)^2 + \sum_{i=k_1+2}^{F} (k_i - 2)^2 \leqq (V - k_1)(V - k_1 - 1), \tag{15}$$

$$\sum_{i=2}^{1+\deg v_1} (\deg v_i - 3)^2 + \sum_{i=2+\deg v_1}^{V} (\deg v_i - 2)^2 \leqq (F - \deg v_1)(F - \deg v_1 - 1), \tag{16}$$

$$\begin{aligned} &\sum_{v_i \in N(v_j)} (\deg v_i - 3)^2 + \sum (\deg v_i - 2)^2 + \sum \deg v_i (\deg v_i - 2) \\ &\leqq (F - \deg v_j)(F - \deg v_j - 1) \qquad \textit{for every } 1 \leqq j \leqq V, \end{aligned} \tag{17}$$

where the second sum is taken over all the vertices v_i such that v_i and v_j lie in the same facet but not in the same edge, and the third sum is taken over all the vertices v_i such that v_i and v_j are not in the same facet.

PROOF. (15) is obtained as follows: (2) implies that $(k_i - 2)^2 \leqq k_i(k_i - 2)$ for every $1 \leqq i \leqq F$. Hence we obtain from (11), noting that $b_j \geqq 0$:

$$\sum_{|f_i \cap f_j| = 2} (k_i - 3)^2 + \sum_{|f_i \cap f_j| = 1} (k_i - 2)^2 \leqq (V - k_j)(V - k_j - 1) \qquad \textit{for every } 1 \leqq j \leqq F.$$

Taking here $j = 1$ and noting that

$$i_1 < i_2 \Rightarrow k_{i_1} \geqq k_{i_2} \Rightarrow (k_{i_1} - 3)^2 + (k_{i_2} - 2)^2 \leqq (k_{i_2} - 3)^2 + (k_{i_1} - 2)^2,$$

(15) follows.

If M is a polyhedral map then its dual M^* is also a polyhedral map. Applying (15) to M^* (where the parameters V, F, etc. refer to M) we get (16). Applying (11) to the map dual to M we get (17), since $b_j \geqq 0$. Note that for w.n.p. maps the third sum is empty. □

The fact that (the boundary complex of) every 3-dimensional pyramid is a w.n.p. map shows that k_1 can be as large as $V - 1$, and the bound given in (4) for k_1 is therefore sharp. However, for $g > 0$ the situation is much different, as shown in the next theorem.

THEOREM 3. *For every polyhedral map with $g > 0$,*

$$k_1 + k_2 - |f_1 \cap f_2| \leqq V - 1, \tag{18}$$

$$k_1 \leqq V - 3. \tag{19}$$

For every w.n.p. map with $g>0$,

(20) $$k_1 \leqq 6 \quad or \quad 2k_1 - 1 \leqq V.$$

PROOF. Let M be a polyhedral map. Then obviously $k_1 + k_2 - |f_1 \cap f_2| \leqq V$. Assume $k_1 + k_2 - |f_1 \cap f_2| = V$. Then $f_1 \cup f_2$ contains all the vertices. We will show that $g = 0$. Since it is always possible to subdivide all facets except f_1, f_2 into triangles without affecting g and V, we can assume w.l.o.g. that all the facets except possibly f_1, f_2 are triangles.

If $f_1 \cap f_2 = \emptyset$ then each of the triangles $f_3, \ldots, f_F$ shares an edge with either f_1 or f_2, thus $F - 2 = k_1 + k_2 = V$. If $|f_1 \cap f_2| \geqq 1$ then there are exactly two triangles within $f_3, \ldots, f_F$ which share an edge with f_1 *and* f_2, the others share an edge with either f_1 or f_2. This implies $F - 2 = k_1 + k_2 - 2|f_1 \cap f_2|$, recalling that $|f_1 \cap f_2| = 2$ means that f_1 and f_2 share an edge. Thus in any case $F - 2 = V - |f_1 \cap f_2|$.

Now Euler's equation for the map M yields $2 - 2g = V - \frac{1}{2}(k_1 + k_2 + 3(F-2)) + F = 2$, thus $g = 0$, which shows (18).

Since $k_2 \geqq 3$ we get from (18) $k_1 \leqq V - 2$. Now assume that M is a polyhedral map with $g > 0$ and $k_1 = V - 2$. Then $k_2 = 3$, and therefore $k_i = 3$ for $2 \leqq i \leqq F$. Now (1) implies $2V + 4g - 4 = V - 5 + F$, that is, $4g = F - V - 1$.

As $k_1 = V - 2$, the map M contains just two vertices x and y that are not in f_1. If x, y are joined by an edge xy in M, then exactly two facets of $f_2, \ldots, f_F$ contain the edge xy and thus no edge of f_1. Thus $F - 1 = k_1 + 2$. Hence $4g = F - V - 1 = 0$, contradicting $g > 0$. If x, y are not joined by an edge we get similarly $F - 1 = k_1$, thus $4g = F - V - 1 = -2$, contradicting $g > 0$. Now (19) is proved.

We turn to the proof of (20). For every $1 \leqq i \leqq F$ we have $0 \leqq (k_i - 4)^2 = k_i^2 - 8k_i + 16$, hence $(k_i - 3)^2 \geqq 2k_i - 7$. Also, $(k_i - 2)^2 \geqq k_i - 2$. Thus we obtain from (15):

$$(V-k_1)(V-k_1-1) \geqq \sum_{i=2}^{k_1+1} (k_i-3)^2 + \sum_{i=k_1+2}^{F} (k_i-2)^2 \geqq \sum_{i=2}^{k_1+1} (2k_i-7) + \sum_{i=k_1+2}^{F} (k_i-2)$$

$$= 2\sum_{i=2}^{k_1+1} k_i - 7k_1 + \sum_{i=1}^{F} (k_i-2) + 2(k_1+1) - \sum_{i=2}^{k_1+1} k_i - k_1$$

$$= \sum_{i=2}^{k_1+1} k_i + \sum_{i=1}^{F} (k_i-2) - 6k_1 + 2 \underset{(1)}{=} \sum_{i=2}^{k_1+1} k_i + 2V + 4g - 4 - 6k_1 + 2$$

$$\underset{(13)}{\geqq} (k_1-2)(V-k_1+2) + 6 - 4g + 2V + 4g - 4 - 6k_1 + 2 = k_1(V-k_1-1) - k_1.$$

Therefore, $(2k_1 - V)(V - k_1 - 1) \leqq k_1$.

Now define $c = 2k_1 - V$. The last inequality implies that $k_1(c-1) \leqq c(c+1)$. If $c \leqq 1$ then $2k_1 - 1 \leqq V$. If $c \geqq 2$ then

$$k_1 \leqq \frac{c(c+1)}{c-1} = c + 2 + \frac{2}{c-1},$$

and this implies that $k_1 \leqq 6$ or $k_1 \leqq c+2$ (and $c \geqq 4$). The last case, namely $k_1 \leqq c+2$, implies $V - k_1 = k_1 - c \leqq 2$, which contradicts (19), hence this is not possible, and (20) is thus proved. □

4. Upper bound for $V(g)$

Our main purpose in the present section is to prove that $V(g)$ — the maximal number of vertices that a w.n.p. map of genus g (orientable or not) can have — is finite for every $g > 0$, and to find a good upper bound for $V(g)$.

In (14) we found a function of V and k_1 which is bounded by $4g$. In Lemma 8 we describe two other such functions. Those three functions are compounded in Theorem 4 to yield a function $f(V, k_1)$ such that $f(V, k_1) \leqq 4g$ for every w.n.p. map with $g > 0$. Lemma 9 enables us to get from this inequality another inequality (Theorem 7) in which a (rather complicated) function of V only is shown to be $\leqq 4g$ for every w.n.p. map with $g > 0$. Since Lemma 9 holds for $V \geqq 64$, Theorem 7 too holds with that restriction. From Theorem 7 we get (Theorem 8) an upper bound for $V(g)$, but we still need the restriction $V \geqq 64$. In order to get rid of that restriction and to get an even better lower bound for $4g$ in terms of V and k_1, we describe in Theorem 5 another function $f_4(V, k_1)$ which is $\leqq 4g$ for every $g > 0$. Theorem 6 combines the functions f and f_4 to yield a function $h(V, k_1)$ which is $\leqq 4g$ for every $g > 0$. In Table 1 we give the values of $h(V, k_1)$ for every $V \leqq 65$ and every k_1 in its permitted domain. From Table 1 we derive (Table 2) upper bounds for $V(g)$ for some small values of g. Table 1 enables us to improve Theorem 7 in the sense that it now holds for every $V \geqq 22$, and Table 2 enables us to extend the bound for $V(g)$ to hold for every $g > 0$. Those improvements are already incorporated in Theorems 7 and 8.

It is clear (e.g. from (1) or (5)) that for every fixed V and g, the number of p-vectors of w.n.p. maps of genus g with V vertices is finite, and, if p is a fixed p-vector, then the number of w.n.p. maps of genus g with V vertices, and having p as a p-vector, is finite. Thus, the boundedness of $V(g)$ implies (Theorem 9) that for every $g > 0$ the number of w.n.p. maps of genus g is finite. Theorem 8 also enables us to describe (Theorem 10) the asymptotic behaviour of $V(g)$ as $g \to \infty$.

TABLE 1

Function $h(V, k_1), 9 \leqq V \leqq 65, 3 \leqq k_1 \leqq \max\{6, \frac{1}{2}(V+1)\}$

65	1261	914	706	568	469	394	337	362	389	412	431	446	462	496	530
64	1220	884	683	548	452	380	324	354	380	402	420	434	452	487	519
63	1180	855	660	529	436	367	317	346	371	392	409	422	443	477	509
62	1141	826	637	511	421	353	310	338	362	382	398	410	434	467	498
61	1102	797	614	492	405	340	303	330	353	372	387	398	425	457	487
60	1064	769	592	474	390	327	296	322	344	362	376	386	416	447	476
59	1027	742	571	457	375	314	289	314	335	352	365	376	407	437	466
58	990	715	550	439	361	302	282	306	326	342	354	368	398	427	455
57	954	688	529	422	346	289	275	298	317	332	343	359	389	417	444
56	919	662	508	406	332	277	268	290	308	322	332	351	380	407	433
55	884	637	488	389	319	266	261	282	299	312	321	343	371	397	422
54	850	612	469	373	305	254	254	274	290	302	310	335	362	387	412
53	817	587	450	356	292	243	247	266	281	292	299	326	353	377	401
52	784	563	431	342	279	232	240	258	272	282	292	318	344	368	390
51	752	540	412	327	267	221	233	250	263	272	284	310	334	358	380
50	721	517	394	313	254	211	226	242	254	262	277	302	326	348	369
49	690	494	377	298	242	200	219	234	245	252	269	294	317	338	358
48	660	472	360	284	231	194	212	226	236	242	262	285	308	328	347
47	631	451	343	271	219	183	205	218	227	232	255	277	298	318	337
46	602	430	326	257	208	182	198	210	218	224	247	269	290	308	326
45	574	409	310	244	197	176	191	202	209	218	240	261	281	299	316
44	547	389	295	232	187	170	184	194	200	212	233	253	272	289	305
43	520	370	280	219	176	164	177	186	191	205	226	245	262	279	295
42	494	351	265	207	166	158	170	178	182	199	218	237	254	269	284
41	469	332	250	196	157	152	163	170	173	192	211	228	245	260	274
40	444	314	236	184	147	146	156	162	166	186	204	220	236	250	263
39	420	297	223	173	138	140	149	154	161	179	196	212	227	241	254
38	397	280	210	163	129	134	142	146	155	173	189	204	218	231	244
37	374	263	197	152	121	128	135	138	149	166	182	196	210	222	233
36	352	247	184	142	112	122	128	130	144	160	175	188	200	213	224
35	331	232	172	133	107	116	121	122	138	154	167	180	192	203	216
34	310	217	161	123	102	110	114	117	132	147	160	172	184	195	206
33	290	202	150	114	97	104	107	112	127	140	153	165	175	187	206
32	271	188	139	106	92	98	100	107	122	134	146	157	168	178	
31	252	175	128	97	87	92	93	103	116	128	139	149	160	178	
30	234	162	118	89	82	86	86	98	110	122	132	142	152		
29	217	149	109	82	77	80	80	93	105	116	125	136	152		
28	200	137	100	74	72	74	76	88	99	110	119	128			
27	184	126	91	67	67	68	72	84	94	103	113	128			
26	169	115	82	61	62	62	68	79	89	98	106				
25	154	104	74	54	57	56	65	74	83	92	106				
24	140	94	67	50	52	50	61	70	78	86					
23	127	85	60	46	47	48	57	65	74	86					
22	114	76	53	42	42	44	53	61	68						
21	102	67	46	38	37	41	49	57	68						
20	91	59	40	34	32	38	46	52							
19	80	52	35	30	28	35	42	52							
18	70	45	30	26	26	32	38								
17	61	38	25	22	23	30	38								
16	52	32	20	18	21	26									
15	44	27	17	14	19	26									
14	37	22	14	12	16										
13	30	17	11	10	16										
12	24	13	8	8											
11	19	10	5	8											
10	14	7	2	6											
9	10	4	2	2											
$k_1 =$	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17

563	593	623	651	678	703	727	749	771	790	810	828	846	363	886	926
551	581	610	637	663	687	710	732	752	772	790	808	826	845	866	
539	569	596	623	647	671	693	715	734	753	770	789	805	827	866	
528	556	583	608	633	655	677	697	716	734	752	769	788	808		
516	543	570	594	618	639	660	679	698	715	733	749	770	808		
504	531	556	580	603	624	644	662	680	697	714	732	752			
493	519	543	566	587	608	627	646	662	680	695	716	752			
481	506	530	552	573	592	611	628	645	662	679	698				
470	494	516	538	558	576	595	611	628	643	663	698				
458	481	503	524	543	561	578	594	611	628	646					
446	469	490	510	528	546	562	578	593	612	646					
434	456	476	496	513	530	546	562	578	596						
423	444	464	482	499	515	531	545	564	596						
412	431	450	468	484	500	515	531	548							
400	419	437	454	470	485	499	517	548							
388	407	424	440	455	470	486	502								
377	395	411	426	442	455	472	502								
366	382	398	413	428	442	458									
354	370	385	400	413	430	458									
342	358	372	387	401	416										
332	346	360	373	389	416										
320	334	348	362	376											
309	323	335	350	376											
298	311	324	338												
287	299	314	338												
276	289	302													
265	279	302													
256	268														
246	268														
236															
236															
18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33

TABLE 2

$V_h(g)$ is the upper bound for $V(g)$ derived from Table 1, and k_1 indicates the value of k_1 at which $V_h(g)$ is attained

g	$\frac{1}{2}-1$	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}-4$	$4\frac{1}{2}$	5	$5\frac{1}{2}$	6	$6\frac{1}{2}$	7	$7\frac{1}{2}$	8
$V_h(g)$	10	11	12	13	14	15	16	16	17	17	18	19	19	20
k_1	5	5	5, 6	6	6	6	6	5, 6	6	6, 7	6, 7	7	6, 7	7

g	$8\frac{1}{2}-9$	$9\frac{1}{2}-10$	$10\frac{1}{2}$	11	$11\frac{1}{2}$	12	$12\frac{1}{2}$	13	$13\frac{1}{2}$	14	$14\frac{1}{2}-15$
$V_h(g)$	20	21	22	22	23	23	24	24	25	25	25
k_1	6, 7	6, 7	6, 7	6, 7, 8	6	6, 7, 8	6, 8	6, 7, 8	6	6, 8	6, 7, 8

LEMMA 8. *The following hold for every w.n.p. map with $g > 0$:*

(21)
$$\left\lceil \frac{V(V-1)}{k_1} \right\rceil - 2V + 4 \leqq 4g,$$

If $k_1(k_1+1) \geqq V$, then

(22)
$$4g \geqq (k_1-3)(V-k_1) - 2nk + n^2 + 2 + \begin{cases} k_1, & n^2 \leqq V - k_1 \leqq n^2+n-1, \\ n, & n^2+n \leqq V-k_1 \leqq n^2+2n. \end{cases}$$

PROOF. From (1) and (5) we get

$$V(V-1) = \sum_{i=1}^{F} k_i(k_i-2) \leqq k_1 \sum_{i=1}^{F} (k_i - 2) = k_1(2V + 4g - 4)$$

and (21) follows.

to prove (22), assume $k_1(k_1+1) \geqq V$. Then $k_1 \geqq \mu$ for both $\mu = \lceil \sqrt{V-k_1} \rceil$ and $\mu = \lfloor \sqrt{V-k_1} \rfloor$. Clearly, $k_1 \leqq F-1$ (as f_1 shares at most one edge with every other f_i), hence $\mu + 1 \leqq k_1 + 1 \leqq F$, and (3) (for $\mu+1$) implies

(*)
$$k_1 + k_2 + \cdots + k_{\mu+1} \leqq V + \mu(\mu+1).$$

Since $i > j \Rightarrow k_i \leqq k_j$, it follows that $\mu k_{\mu+1} \leqq k_2 + \cdots + k_{\mu+1} \leqq V - k_1 + \mu(\mu+1)$, hence

(**)
$$k_i \leqq k_{\mu+1} \leqq \left\lfloor \frac{V-k_1}{\mu} \right\rfloor + \mu + 1 \qquad \text{for every } \mu + 1 \leqq i \leqq F.$$

From (*) and (**) we get

$$\begin{aligned} k_1 + k_2 + \cdots + k_{k_1+1} &\leqq V + \mu(\mu+1) + (k_1 - \mu)\left(\left\lfloor \frac{V-k_1}{\mu} \right\rfloor + \mu + 1\right) \\ &= V + k_1(\mu+1) + (k_1-\mu)\left\lfloor \frac{V-k_1}{\mu} \right\rfloor \end{aligned}$$

and using (13) we get

$$(k_1-2)(V-k_1+2)+6-4g \leqq k_2+\cdots+k_{k_1+1} \leqq V+k_1\mu+(k_1-\mu)\left\lfloor \frac{V-k_1}{\mu} \right\rfloor,$$

that is

$$4g \geqq (k_1-3)(V-k_1)-k_1(\mu-1)-(k_1-\mu)\left\lfloor \frac{V-k_1}{\mu} \right\rfloor+2.$$

If $n^2 \leqq V-k_1 \leqq n^2+n-1$ we choose $\mu=n=\lfloor\sqrt{V-k_1}\rfloor$, and if $n^2+n \leqq V-k_1 \leqq n^2+2n$ we take $\mu=n+1=\lceil\sqrt{V-k_1}\rceil$, and we get (22). □

We now combine inequalities (14), (21) and (22) into a single theorem. For that we define

$$f_1(V,k_1)=\left\lceil \frac{V(V-1)}{k_1} \right\rceil -2V+4, \qquad f_2(V,k_1)=(k_1-2)(V-2k_1)+2,$$

$$f_3(V,k_1)=(k_1-3)(V-k_1)-2nk_1+n^2+2+\begin{cases} k_1, & n^2 \leqq V-k_1 \leqq n^2+n-1, \\ n, & n^2+n \leqq V-k_1 \leqq n^2+2n. \end{cases}$$

Note that $f_1(V,k_1)=f_2(V,k_1)$ for $V=(k_1-1)^2$ and $f_2(V,k_1)=f_3(V,k_1)$ for $V=\frac{1}{4}(k_1+1)^2$. Also $V \leqq \frac{1}{4}(k_1+1)^2$ implies $k_1(k_1+1) \geqq V$. Hence (14), (21) and (22) yield:

THEOREM 4. *Define*

$$f(V,k_1)=\begin{cases} f_1(V,k_1) & \text{for } V \geqq (k_1-1)^2, \\ f_2(V,k_1) & \text{for } (k_1-1)^2 \geqq V \geqq \frac{1}{4}(k_1+1)^2, \\ f_3(V,k_1) & \text{for } \frac{1}{4}(k_1+1)^2 \geqq V. \end{cases}$$

Then $f(V,k_1) \leqq 4g$ for every w.n.p. map with $g>0$.

REMARK. We could have shown that $f(V,k_1)=\max\{f_1(V,k_1), f_2(V,k_1), f_3(V,k_1)\}$ (this, indeed, motivates the definition of f), but we will not need it.

In the next theorem we present another function of V and k_1, which, like the previous ones, is bounded by $4g$. (Recall from Theorem 3 that $k_1 \leqq V-3$ and $2k_1-1 \leqq V$ for $g>0$.)

THEOREM 5. *Let*

$$f_4(V,k_1)=k_1(V-k_1)-2V+2 - \left\lfloor \frac{1}{2\lambda+1}((V-2k_1+1)(V-k_1-2)+2+(\lambda^2+\lambda)k_1) \right\rfloor$$

where $\lambda=\lfloor k_1^{-1/2}((V-2k_1+1)(V-k_1-2)+2)^{1/2} \rfloor$.

Then $f_4(V, k_1) \leqq 4g$ *for every w.n.p. map with* $g > 0$.

PROOF. Let μ be any integer $\geqq 0$. For every $1 \leqq i \leqq F$, $k_i - \mu - 4$ and $k_i - \mu - 5$ are two integers which differ by 1, hence their product is $\geqq 0$. So

$$0 \leqq (k_i - \mu - 4)(k_i - \mu - 5) = k_i^2 - 2\mu k_i - 9k_i + \mu^2 + 9\mu + 20,$$

hence $(k_i - 3)^2 \geqq (2\mu + 3)k_i - \mu^2 - 9\mu - 11$. Also $(k_i - 2)^2 \geqq k_i - 2$. Thus we obtain from (15)

$$\begin{aligned}(V - k_1)(V - k_1 - 1) &\geqq \sum_{i=2}^{k_1+1} ((2\mu + 3)k_i - \mu^2 - 9\mu - 11) + \sum_{i=k_1+2}^{F} (k_i - 2)\\ &= (2\mu + 2)\sum_{i=2}^{k_1+1} k_i + \sum_{i=1}^{F}(k_i - 2) - (k_1 - 2) - (\mu^2 + 9\mu + 9)k_1\\ &\underset{(1),(13)}{\geqq} (2\mu + 2)((k_1 - 2)(V - k_1 + 2) + 6 - 4g) + 2V + 4g - 2\\ &\quad - (\mu^2 + 9\mu + 10)k_1.\end{aligned}$$

From this we get by simple calculation:

$$4g \geqq k_1(V - k_1) - 2V + 2 - \frac{1}{2\mu + 1}((V - 2k_1 + 1)(V - k_1 - 2) + 2 + (\mu^2 + \mu)k_1).$$

This already proves our theorem, since $4g$ and $\mu = \lambda$ are integers. However, we are going to show that the choice $\mu = \lambda$ is the best choice for μ, in the sense that it minimizes the last term in the right side of the last inequality. For that, define

$$t = (V - 2k_1 + 1)(V - k_1 - 2) + 2, \qquad A(\beta) = \frac{1}{2\beta + 1}(t + (\beta^2 + \beta)k_1).$$

The function $A(\beta)$, for β *real*, has a unique minimum, attained at $\beta_0 = -\frac{1}{2} + \frac{1}{2}\sqrt{(4t - k_1)/k_1}$. Hence the minimal value of $A(\mu)$, for μ *integral*, is attained either at $\mu = \lfloor \beta_0 \rfloor$ or at $\mu = \lceil \beta_0 \rceil$. However, it is not easy to see from this that the choice $\mu = \lambda$, as stated in our theorem, is a unification of these two possibilities.

But we are looking for an *integer* μ which will minimize the function $\lfloor A(\mu) \rfloor$. Clearly $a \leqq b \Rightarrow \lfloor a \rfloor \leqq \lfloor b \rfloor$, hence it is sufficient to minimize $A(\mu)$. $A(\mu)$ for μ real is strictly decreasing for $0 \leqq \mu \leqq \beta_0$ and strictly increasing for $\beta_0 \leqq \mu$. Thus $A(\mu)$ for $\mu \geqq 0$ integral is minimized at the largest integer μ such that $A(\mu) \leqq A(\mu - 1)$. Simplifying $A(\mu) \leqq A(\mu - 1)$ we get $\mu^2 \leqq t/k_1$, hence the integer μ which minimizes $A(\mu)$ is $\mu = \lambda = \lfloor \sqrt{t/k_1} \rfloor$. □

Combining the last two theorems we get

THOEREM 6. *Let* $h(V, k_1) = \max\{f(V, k_1), f_4(V, k_1)\}$. *Then* $h(V, k_1) \leqq 4g$ *holds for every w.n.p. map with* $g > 0$.

Recall that $V(g)$ is the maximal V such that there exists a w.n.p. map of genus g (orientable or not) with V vertices ($V(g) = \infty$ if there is no maximal V, $V(g) = 0$ if there is no w.n.p. map of genus g), and $V(g^+)$ ($V(g^-)$) stands similarly for orientable (resp. non-orientable) maps. For every $g \in \{\frac{1}{2}n : n \in \mathbf{N}\}$ ($\mathbf{N}$ is the set of natural numbers) define

$$V_h(g) = \sup\{V : h(V, k_1) \leqq 4g,\ 3 \leqq k_1 \leqq \max\{6, \tfrac{1}{3}(V+1)\},\ V, k_1 \in \mathbf{N}\}.$$

From Theorems 3, 6 it follows that $V(g) \leqq V_h(g)$ for every $g > 0$. Later (Theorem 8) we will see that $V_h(g) < \infty$ for every $g > 0$, and we estimate $V_h(g)$ for every $g \geqq 81$. In order to facilitate the calculation of $V_h(g)$ for smaller values of g, we give in Table 1 the values of $h(V, k_1)$ for $9 \leqq V \leqq 65$, $3 \leqq k_1 \leqq \max\{6, \frac{1}{3}(V+1)\}$. (The last restriction on k_1 follows from Theorem 3.)

From Table 1 we see that $h(10, 5) = 2$, while $h(V, k_1) > 4$ for every $V > 10$ and every k_1 in the permitted domain. Therefore it follows that $V_h(\frac{1}{2}) = V_h(1) = 10$ and hence $V(\frac{1}{2}), V(1) \leqq 10$. Moreover, for every $k_1 \neq 5$ in the permitted domain we have $h(10, k_1) > 4$, hence $V_h(\frac{1}{2})$ and $V_h(1)$ are attained at $k_1 = 5$ only. Thus we conclude that there are no w.n.p. maps on the projective plane, on the torus and on the Klein bottle with more than 10 vertices, and if there is such a w.n.p. map with 10 vertices, then in that map the largest facet has precisely 5 vertices. In fact, we know that $V(\frac{1}{2}) = 10$ (see [7]), $V(1^+) = 9$ (see [8]) and $V(1^-) = 10$ (see [7]), thus the bound 10 for $V(\frac{1}{2})$ and $V(1)$ is indeed sharp.

Similarly we see from Table 1 that $V_h(2) = 12$, and that $V_h(2)$ is attained only at $k_1 = 5$ and at $k_1 = 6$. Thus $V(2) \leqq 12$. Surprisingly, the detailed investigation carried on in [3] shows that there exists no *orientable* w.n.p. map of genus 2 at all, hence $V(2^+) = 0$. We can prove that $V(2^-) = 10$ (see [5]), hence the bound 12 for $V(2)$ is not sharp. Also for $g = \frac{3}{2}$, Table 1 implies $V(\frac{3}{2}) \leqq 11$, while (see [4]) the correct value is $V(\frac{3}{2}) = 10$.

In the same manner we can read from Table 1 the values of $V_h(g)$ for every $0 < g \leqq 84$, and for every such g we can read from Table 1 the values of k_1 at which $V_h(g)$ is attained. In Table 2 we give that information for $\frac{1}{2} \leqq g \leqq 15$.

REMARK. In the discussion which follows we will be using only the function $f(V, k_1)$, and no further information will be drawn from $f_4(V, k_1)$. One might therefore doubt if we need the function $f_4(V, k_1)$ at all, that is, whether we could not give up Theorems 5 and 6, and use the function f instead of h. That this is not the case one can see already from the fact that the analogue of Table 1 for f would have yielded that the minimal V such that $f(V,k_1)>4$ for every k_1 in the permitted domain is $V=14$ (while $f(13,7)=4$), and therefore such a table for f would have yielded the bound 13 for $V(1)$.

LEMMA 9. *For every $V\geqq 64$ the minimal value of $f(V,k_1)$ is attained either at $k_1=\lfloor\sqrt{V}\rfloor+1$ (and then $f(V,k_1)=f_1(V,k_1)$) or at $k_1=\lceil\sqrt{V}\rceil+1$ (and then $f(V,k_1)=f_2(V,k_1)$).*

PROOF. Fix $V\geqq 64$. Then $f(V,k_1)$ is a function of k_1, and its domain (for k_1 real) divides naturally into three parts (recall from Theorem 3 that $2k_1-1\leqq V$):

$$\mathrm{I}_1\colon [3;\sqrt{V}+1],\qquad \mathrm{I}_2\colon [\sqrt{V}+1,2\sqrt{V}-1],\qquad \mathrm{I}_3\colon \left[2\sqrt{V}-1;\frac{V+1}{2}\right],$$

and for $1\leqq i\leqq 3$, $f(V,k_1)=f_i(V,k_1)$ in I_i. First we show that the minimum of $f(V,k_1)$ is not attained in I_3.

Since $g>0$, Lemma 9 implies $V-k_1\geqq 3$. Hence:

$$2n-1\leqq 2\sqrt{V-k_1+1}-1 \qquad \text{for } n^2\leqq V-k_1\leqq n^2+n-1,$$

$$2n\leqq 2\sqrt{V-k_1+1}-1 \qquad \text{for } n^2+n\leqq V-k_1\leqq n^2+2n,$$

$$n^2\geqq V-k_1-\sqrt{V-k_1+1} \qquad \text{for } n^2\leqq V-k_1\leqq n^2+n-1,$$

$$n^2+n\geqq V-k_1-\sqrt{V-k_1+1} \qquad \text{for } n^2+n\leqq V-k_1\leqq n^2+2n.$$

From these, from (22) and from the definition of $f_3(V,k_1)$ we get

$$\begin{aligned}4g&\geqq f_3(V,k_1)\\ &\geqq (k_1-3)(V-k_1)+2-(2\sqrt{V-k_1+1}-1)k_1+V-k_1-\sqrt{V-k_1+1}\\ &=(k_1-2)(V-k_1)-(2k_1+1)\sqrt{V-k_1+1}+k_1+2\overset{\text{def}}{=}A(V,k_1).\end{aligned}$$

$A(V,k_1)$, as a function of k_1 with parameter V, is easily seen to be concave in I_3, hence it attains its minimal value at an endpoint of I_3. Simple calculation yields that

$$A\left(V,\frac{V+1}{2}\right)\geqq \tfrac14 V^2-\tfrac12 V+\tfrac{13}{4}-(V+2)\sqrt{V},$$

$$A(V,2\sqrt{V}-1)\leqq 2V\sqrt{V}-11V+15\sqrt{V}-3,$$

and $2V\sqrt{V}-11V+15\sqrt{V}-3\leqq\frac14 V^2-\frac12 V+\frac{13}{4}-(V+2)\sqrt{V}$ for $V\geqq 64$.

(To prove the last inequality, denote $\alpha = \sqrt{V}$. We have to show that $\alpha^4 - 12\alpha^3 + 42\alpha^2 - 68\alpha + 25 \geqq 0$ for $\alpha \geqq 8$. Indeed, for $\alpha \geqq 8$ we have $\alpha^4 - 12\alpha^3 + 42\alpha^2 - 68\alpha + 25 = (\alpha^2 - 6\alpha + 3)^2 - 32\alpha + 16 \geqq 19(\alpha^2 - 6\alpha + 3) - 32\alpha + 16 > 0$.)

Now, comparing $\frac{1}{4}V^2 - \frac{1}{2}V + \frac{13}{4} - (V+2)\sqrt{V}$ to the value of $f_2(V, k_1)$ at $k_1 = \lceil\sqrt{V}\rceil + 1$ we see that

$$f_2(V, \lceil\sqrt{V}\rceil + 1) = (\lceil\sqrt{V}\rceil - 1)(V - 2\lceil\sqrt{V}\rceil - 2) + 2 \leqq \sqrt{V}(V - 2\sqrt{V} - 2) + 2$$
$$(*) \qquad = V\sqrt{V} - 2V - 2\sqrt{V} + 2$$

and $V\sqrt{V} - 2V - 2\sqrt{V} + 2 \leqq \frac{1}{4}V^2 - \frac{1}{2}V + \frac{13}{4} - (V+2)\sqrt{V}$ for $V \geqq 64$.

(To prove the last inequality, let $\alpha = \sqrt{V}$. We have to show $\alpha^4 - 8\alpha^3 + 6\alpha^2 + 5 \geqq 0$ for $\alpha \geqq 8$. For $\alpha = 8$ this inequality holds, and the left side, as a function of α, is increasing for $\alpha \geqq 6$.) Thus the minimal value of $f(V, k_1)$ as a function of k_1 with parameter V is not attained in I_3, for $V \geqq 64$.

The function $f_2(V, k_1)$ is concave in the interval I_2, hence it attains its minimum at an endpoint of I_2, and for k_1 integer — at an endpoint of $I_2 \cap \mathbf{N}$ ($\mathbf{N}$ = the natural numbers). Now

$$f_2(V, \lfloor 2\sqrt{V}\rfloor - 1) = (\lfloor 2\sqrt{V}\rfloor - 3)(V - 2\lfloor 2\sqrt{V}\rfloor + 2) + 2$$
$$\geqq (2\sqrt{V} - 4)(V - 4\sqrt{V} + 2) + 2$$
$$= 2V\sqrt{V} - 12V + 20\sqrt{V} - 8$$

and, since $V\sqrt{V} - 2V - 2\sqrt{V} + 2 \leqq 2V\sqrt{V} - 12V + 20\sqrt{V} - 8$ for $V \geqq 64$ (denote $\alpha = \sqrt{V}$ and proceed as before), we get from (*) that for $V \geqq 64$, $f_2(V, k_1)$ attains its minimum in $I_2 \cap \mathbf{N}$ at $k_1 = \lceil\sqrt{V}\rceil + 1$.

$f_1(V, k_1)$ is a decreasing function of k_1 with parameter V, hence its minimum in $I_1 \cap \mathbf{N}$ is attained at the endpoint $k_1 = \lfloor\sqrt{V}\rfloor + 1$ of $I_1 \cap \mathbf{N}$. This concludes the proof of the lemma. □

REMARK. Lemma 9 holds already for all $V \geqq 28$, as we had observed by direct computations of all the cases $28 \leqq V \leqq 65$. For $V = 30$ (28) that minimum is attained, not only at the value of k_1 indicated in the lemma, but also at $k_1 = 10$ (resp. 9).

Direct computations show that the minimal value of $h(V, k_1)$ coincides with that of f for all V, $28 \leqq V \leqq 65$, and is attained precisely at the points indicated in Lemma 9. Moreover, the minimal value of $h(V, k_1)$ is attained at the points indicated in Lemma 9 already for $V \geqq 22$. (For $V = 24$ that minimum (50) is attained, not only at $k_1 = 6$, but also at $k_1 = 8$.) In other words, if in Lemma 9 we replace f by h, then Table 1 reflects that the new statement is correct for every $22 \leqq V \leqq 65$.

Combining Theorem 6 with Lemma 9 and the last remark we get:

THEOREM 7. *For every w.n.p. map with $g > 0$ and $V \geqq 22$ we have*

$$4g \geqq h(V, k_1) \geqq \min\left\{\left\lceil \frac{V(V-1)}{\lfloor\sqrt{V}\rfloor + 1}\right\rceil - 2V + 4; (\lceil\sqrt{V}\rceil - 1)(V - 2\lceil\sqrt{V}\rceil - 2) + 2\right\}. \tag{23}$$

Finally we get the main result of this section.

THEOREM 8. *For every $g > 0$ of the form $\frac{1}{2}n$, $n \in \mathbf{N}$, define*

$$\bar{V}(g) = \max\left\{V \in \mathbf{N} : \min\left\{\left\lceil \frac{V(V-1)}{\lfloor\sqrt{V}\rfloor + 1}\right\rceil - 2V + 4; (\lceil\sqrt{V}\rceil - 1)(V - 2\lceil\sqrt{V}\rceil - 2) + 2\right\} \leqq 4g\right\}.$$

Then

(a) $\bar{V}(g) < \infty$ *for every* $g > 0$,

(b) $V(g) \leqq \bar{V}(g)$ *for every* $g > 0$ *with the exception of* $g = 2\frac{1}{2}, 3, 3\frac{1}{2}, 4, 4\frac{1}{2}, 7, 8$, *and for these exceptions* $V(g) \leqq \bar{V}(g) + 1$.

PROOF. Theorem 6 implies that $V(g) \leqq V_h(g)$ for every $g > 0$. The right side of (23) is a strictly increasing function of V whose values are integers, and its value for $V = 22$ is $42 = 4 \cdot 10\frac{1}{2}$. Hence $\bar{V}(g) < \infty$ for every $g > 0$, and, using Theorem 5, $V_h(g) \leqq \bar{V}(g)$ for every $g > 10$. The results for the exceptional values of g follow from Tables 1 and 2. □

REMARK. It can be proved that $V_h(g) = \bar{V}(g)$ for all $g > 0$ with the exception of the values given in Theorem 8(b), and for those exceptions $V_h(g) = \bar{V}(g) + 1$.

Using the remarks at the beginning of the present section we get from Theorem 8:

THEOREM 9. *For every $g > 0$, the number of w.n.p. maps of genus g is finite.*

For every V which is a square ($V = (k_1 - 1)^2$) the two expressions in the right side of (23) are equal to each other, and their common value is $V\sqrt{V} - 3V + 4$. Hence:

LEMMA 10. *For every w.n.p. map with genus $g > 0$ and $V \geqq 22$ which is a square, we have* $4g \geqq V\sqrt{V} - 3V + 4$.

As a corollary of Theorem 8 we get:

THEOREM 10. $\overline{\lim}_{g\to\infty}(V(g)/g^{2/3}) \leqq 2\sqrt[3]{2}$.

An upper bound for g in terms of V and k_1 can be obtained by taking in (6) $k_i = 3$ for all i but $i = 1$ and using (2). We get

$$2g \leqq \left\lfloor \frac{(V-1)(V-6)-k_1(k_1-5)}{6} \right\rfloor \quad \text{for every w.n.p. map.} \tag{24}$$

Since (5) and (6) hold with $\geqq$ instead of $=$ for every polyhedral map, (24) holds even for every polyhedral map.

REMARK. The mere fact that $V(g)$ is bounded for every $g > 0$ (and hence also Theorem 9) could have been deduced already from (1), (2), (3), (5) and from the fact that $k_1 \leqq V-2$ for $g > 0$ (which is less than (19)). Theorem 10 can also be proved using only Theorems 3 and 4. So Theorems 6–9 and Lemma 9 are not necessary for proving Theorem 10.

REFERENCES

1. A. Altshuler, *Polyhedral realization in R^3 of triangulations of the torus and 2-manifolds in cyclic 4-polytopes*, Discrete Math. **1** (1971), 211–238.

2. A. Altshuler and U. Brehm, *A non-Schlegelian polyhedral map on the torus*, Mathematika **31** (1984), 83–88.

3. A. Altshuler and U. Brehm, *Non-existence of weakly neighborly polyhedral maps on the orientable 2-manifold of genus 2*, J. Combinatorial Theory, Ser. A, to appear.

4. A. Altshuler and U. Brehm, *The weakly neighborly polyhedral maps on the 2-manifold with Euler characteristic* −1, Discrete & Computational Geometry, to appear.

5. A. Altshuler and U. Brehm, *The weakly neighborly polyhedral maps on the non-orientable 2-manifold with Euler characteristic* −2, to appear.

6. D. Barnette, *Polyhedral maps on 2-manifolds*, in *Convexity and Related Combinatorial Geometry* (Proc. Second Univ. of Oklahoma Conf., D. C. Kay and M. Breen, eds.), Marcel Dekker Inc., 1982, pp. 7–19.

7. U. Brehm, *Weakly neighborly polyhedral maps on the projective plane, the Möbius strip and the Klein bottle*, to appear.

8. U. Brehm and A. Altshuler, *The weakly neighborly polyhedral maps on the torus*, Geom. Dedic. **18** (1985), 227–238.

9. H. S. M. Coxeter and W. O. J. Moser, *Generators and Relations for Discrete Groups*, 2nd edn., Springer-Verlag, 1965.

10. A. Császár, *A polyhedron without diagonals*, Acta Sci. Math. Szeged **13** (1949–1950), 140–142.

11. B. Grünbaum, *Convex Polytopes*, Wiley, New York, 1967.

12. G. Ringel, *Map Color Theorem*, Springer-Verlag, 1974.

13. J. Simutis, *Geometric Realizations of Toroidal Maps*, Ph.D. Thesis, Univ. of California, Davis, 1977.

14. E. Steinitz and H. Rademacher, *Vorlesungen Uber die Theorie der Polyeder*, Berlin, 1934.